

Introduction

An equation of the Fokker-Planck type is widely used to describe the kinetics of nonequilibrium processes in gases. When this equation is applied to systems with considerable gradients of distribution functions and of transition instants an additional estimate is required of the conditions under which the linear integrodifferential equation is approximated by an equation of the Fokker-Planck type. The replacement of the corresponding integral operator by a differential one was analyzed in many articles (see, for example, [1-3]); the finite expansion was then valid for the distribution function as well as an estimate of the order of magnitude of the terms which follow the transition instants. For strongly nonequilibrium processes and for the time instants  $t < \tau_r$  ( $\tau_r$  is the characteristic relaxation time) the degree of deviation from the equilibrium should be included in the estimate of approximation conditions.

In the present article the exchange is described of the integral operator by a differential one; the operators not being for the distribution function itself but for the ratio to the equilibrium value (the relaxation of the initial distribution being in the form of a  $\delta$ -function is not analyzed). In a number of cases such an expansion enables one at the nonequilibrium or near-equilibrium stages to reduce the sought approximation conditions to the conditions for the transition instants (this is especially suitable if the transition instants are evaluated without using the explicit form for the transition instants), and contains transition instants of only the even order since their evaluation is more straightforward than the evaluation for instants of any order; finally, there is a more obvious possibility of estimating the order of magnitude of the neglected terms and of the accuracy needed for the retained terms. The approach used in [4] was applied to obtain the relevant expansion.

1. Transition to Differential Equation. Expression for Divergence Flow. General Flow of the Approximation Conditions

The starting kinetic equation, similarly as in [4], is written in the form

$$\frac{\partial f}{\partial t} = - \int f(x, t) W(x, \Delta) d\Delta + \int f(x + \Delta, t) W(x + \Delta, -\Delta) d\Delta, \quad (1.1)$$

where  $f(x, t)$  is the distribution function of the molecules with respect to  $x$  (either the energy or the momentum corresponding to the specified degree of freedom) in the gas, playing the part of a thermostat with temperature  $T$ ; the gas-thermostat is of much higher concentration than that of the molecules;  $\Delta$  is the change of  $x$  due to collision;  $W(x, \Delta)$  is the transition probability for the molecule from the state  $(x)$  to  $(x + \Delta)$  in a unit of time. The following notation is now introduced:

$$B_n = \frac{\langle \Delta^n \rangle}{\tau} = \int \Delta^n W(x, \Delta) d\Delta, \quad \frac{1}{\tau} = \int W(x, \Delta) d\Delta; \quad (1.2)$$

$$\varphi(x, t) = f(x, t)/f^0(x), \quad \psi(x, t) = \partial \varphi / \partial x; \quad (1.3)$$

$$\omega(x, \Delta) = f^0(x)W(x, \Delta), \quad (1.4)$$

where  $f^0(x)$  is the equilibrium distribution function corresponding to the temperature  $T(t \rightarrow \infty)$ ;  $B_n$  is the transition instant of order  $n$ .

It follows from the principle of comprehensive balance that

$$\omega(x, \Delta) = \omega(x + \Delta, -\Delta), \quad \omega(x, -\Delta) = \omega(x - \Delta, \Delta). \quad (1.5)$$

By using (1.5), Eq. (1.1) can be written as

$$\frac{\partial f}{\partial t} = \int [\varphi(x + \Delta, t) - \varphi(x, t)] \omega(x, \Delta) d\Delta. \quad (1.6)$$

From (1.6) and taking into account (1.2)-(1.5) and having expanded  $\varphi(x + \Delta, t)$  into a series one finds

$$\frac{\partial f}{\partial t} = \sum_{n=1}^{\infty} \frac{1}{n!} B_n f^0 \frac{\partial^n \varphi}{\partial x^n} = \sum_{n=1}^{\infty} \frac{1}{(2n)!} B_{2n} f^0 \frac{\partial^{2n-1} \psi}{\partial x^{2n-1}} + \sum_{n=1}^{\infty} \frac{1}{(2n-1)!} B_{2n-1} f^0 \frac{\partial^{2n-2} \psi}{\partial x^{2n-2}}. \quad (1.7)$$

The quantity  $B_{2n-1} f^0$  is now expressed in terms of  $B_{2n} f^0$ . Having expanded the right-hand sides of (1.5) into a series of the powers of  $\Delta$ , appearing in the first argument, one obtains

$$\sum_{k=1}^{\infty} \frac{1}{k!} \Delta^k \frac{d^k \omega(x, -\Delta)}{dx^k} = - \sum_{k=1}^{\infty} \frac{1}{k!} (-\Delta)^k \frac{d^k \omega(x, \Delta)}{dx^k}; \quad (1.8)$$

$$\omega(x, \Delta) = \frac{1}{2} [\omega(x, \Delta) + \omega(x, -\Delta)] + \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{k!} \Delta^k \frac{d^k \omega(x, -\Delta)}{dx^k}. \quad (1.9)$$

It follows from (1.9) by taking into account (1.8) that

$$\omega(x, \Delta) = \frac{1}{2} [\omega(x, \Delta) + \omega(x, -\Delta)] - \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{k!} (-1)^k \Delta^k \frac{d^k \omega(x, \Delta)}{dx^k}. \quad (1.10)$$

Employing (1.10) together with (1.2) one finds

$$\begin{aligned} B_{2n-1} f^0 &= \int \Delta^{2n-1} \omega(x, \Delta) d\Delta = - \frac{1}{2} \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \frac{d^k}{dx^k} \times \\ &\times \int \Delta^{2n-1+k} \omega(x, \Delta) d\Delta = \frac{1}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} \frac{d^k}{dx^k} (B_{2n-1+k} f^0) \end{aligned} \quad (1.11)$$

[the result (1.11) is due to the fact that integration takes place over symmetrical intervals]. Then the lower summation limit increases step by step,

$$\begin{aligned} B_{2n-1} f^0 &= \frac{1}{2} \frac{d}{dx} (B_{2n} f^0) + \frac{1}{2} \sum_{k=2}^{\infty} (-1)^{k-1} \frac{1}{k!} \frac{d^k}{dx^k} (B_{2n-1+k} f^0) \equiv \\ &\equiv \frac{1}{2} \frac{d}{dx} (B_{2n} f^0) + \frac{1}{2} \sum_{k=2}^{\infty} (-1)^{k-1} C_{k2} \frac{d^k}{dx^k} (B_{2n-1+k} f^0); \\ B_{2n-1} f^0 &= \frac{1}{2} \frac{d}{dx} (B_{2n} f^0) + \frac{1}{2} \left( C_{32} - \frac{1}{2} C_{22} \right) \frac{d^3}{dx^3} (B_{2n+2} f^0) + \\ &+ \frac{1}{2} \sum_{k=4}^{\infty} \left[ C_{k2} - \frac{1}{2} \frac{C_{22}}{(k-2)!} \right] (-1)^{k-1} \frac{d^k}{dx^k} (B_{2n-1+k} f^0) \equiv \\ &\equiv \frac{1}{2} \frac{d}{dx} (B_{2n} f^0) + \frac{1}{2} \left( C_{32} - \frac{1}{2} C_{22} \right) \frac{d^3}{dx^3} (B_{2n+2} f^0) + \frac{1}{2} \sum_{k=4}^{\infty} (-1)^{k-1} C_{k4} \frac{d^k}{dx^k} (B_{2n-1+k} f^0), \end{aligned}$$

and the procedure can be continued. In this way one obtains

$$B_{2n-1} f^0 = \frac{1}{2} \frac{d}{dx} (B_{2n} f^0) + \sum_{s=1}^{\infty} A_{2s} \frac{d^{2s+1}}{dx^{2s+1}} (B_{2n+2s} f^0); \quad (1.12)$$

$$A_{2s} = \frac{1}{2} \left( C_{2s+1, 2s} - \frac{1}{2} C_{2s, 2s} \right), \quad (1.13)$$

where

$$C_{m2} = \frac{1}{m!}; \quad C_{2m, 2k} = C_{2m, 2k-2} - \frac{1}{2} \frac{C_{2k-2, 2k-2}}{(2m-2k+2)!}; \quad (1.14)$$

$$C_{2m+1, 2k} = C_{2m+1, 2k-2} - \frac{1}{2} \frac{C_{2k-2, 2k-2}}{(2m-2k+3)!}.$$

The coefficients  $C_{2m+1, 2m}$  and  $C_{2m, 2m}$  are related by

$$C_{2m+1, 2m} = \frac{m}{2m+1} C_{2m, 2m}. \quad (1.15)$$

Using (1.13)-(1.15) one obtains from (1.12) that

$$B_{2n-1} = \frac{1}{f^0} \sum_{s=0}^{2n-1} A_{2s} \frac{d^{2s+1}}{dx^{2s+1}} (B_{2n+2s} f^0), \quad (1.16)$$

where

$$A_0 = \frac{1}{2}; \quad A_{2s} = -\frac{1}{4(2s+1)} C_{2s,2s};$$

for  $C_{2s,2s}$  one has the recurrence formula (1.14). The values of the first coefficients  $A_{2s}$  are given by

$$A_0 = \frac{1}{2}, \quad A_2 = -\frac{1}{4!}, \quad A_4 = \frac{3}{6!}, \quad A_6 = -\frac{17}{8!}, \quad A_8 = \frac{155}{10!}.$$

The relation (1.16) enables one to express the transition instant of an odd order in terms of the instants of higher even orders; in particular, it follows from the latter that  $B_{2n-1}$  and  $B_{2n}$  are of the same order of magnitude (excluding singular conditions).

By using (1.16) one can transform Eq. (1.7) into a differential equation of the order  $2n$  which is equivalent to (1.1):

$$\frac{\partial f}{\partial t} = \sum_{n=1} \left[ \frac{1}{(2n)!} B_{2n} f^0 \frac{\partial^{2n-1} \psi}{\partial x^{2n-1}} + \sum_{s=0}^{n-1} \frac{A_{2s}}{(2n-2s-1)!} \frac{\partial^{2n-2s-2} \psi}{\partial x^{2n-2s-2}} \frac{d^{2s+1}}{dx^{2s+1}} (B_{2n} f^0) \right]. \quad (1.17)$$

If all terms with  $n \geq 2$  are neglected in (1.17), then an equation of the Fokker-Planck type is obtained. However, to determine the approximation conditions it is more convenient to start with the equation in its divergence form.

After some transformations which are omitted here (1.17) can be reduced to

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial x} \sum_{n=1} \frac{1}{(2n)!} B_{2n} f^0 \frac{\partial^{2n-2} \psi}{\partial x^{2n-2}} + \frac{\partial}{\partial x} \sum_{n=2} \sum_{k=2}^{2n-1} D_{nk} \frac{\partial^{2n-k-1} \psi}{\partial x^{2n-k-1}} \frac{d^{k-1}}{dx^{k-1}} (B_{2n} f^0); \quad (1.18)$$

$$D_{n,2k} = -D_{n,2k+1}, \quad D_{nk} = (-1)^k \left( -\frac{1}{(2n)!} + \sum_{s=0}^{\left[ \frac{k}{2} \right] - 1} \frac{A_{2s}}{(2n-2s-1)!} \right), \quad \left[ \frac{k}{2} \right] - \text{is an integral part of } \frac{k}{2}. \quad (1.19)$$

By using (1.18) and (1.19) the starting integrodifferential equation (1.1) can be written in the divergence form

$$\frac{\partial f}{\partial t} = -\text{div } j; \quad (1.20)$$

$$-j = G_1 + \sum_{n=2} \left\{ G_n + \sum_{m=1}^{n-1} H_{n,2m} \right\}; \quad (1.21)$$

$$G_n = D_{n1} \frac{\partial^{2n-2} \psi}{\partial x^{2n-2}} B_{2n} f^0; \quad (1.22)$$

$$H_{n,2m} = D_{n,2m} \left[ \frac{\partial^{2n-m-1} \psi}{\partial x^{2n-2m-1}} \frac{d^{2m-1}}{dx^{2m-1}} (B_{2n} f^0) - \frac{\partial^{2n-2m-2} \psi}{\partial x^{2n-2m-2}} \frac{d^{2m}}{dx^{2m}} (B_{2n} f^0) \right]; \quad (1.23)$$

$$D_{n1} = \frac{1}{(2n)!}, \quad D_{n0} = -D_{n1}, \quad D_{n,2m} = D_{n,2m-2} + \frac{A_{2m-2}}{(2n-2m+1)!} = -\frac{1}{(2n)!} + \sum_{s=0}^{m-1} \frac{A_{2s}}{(2n-2s-1)!}. \quad (1.24)$$

It follows from (1.20) and (1.21) that a kinetic equation changes into an equation of the Fokker-Planck type if in the expression for the divergence flow one term only is retained,  $G_1$ . This requires that the condition

$$|G_1| \gg \left| \sum_{n=2} \left\{ G_n + \sum_{m=1}^{n-1} H_{n,2m} \right\} \right| \quad (1.25)$$

be satisfied, and it is sufficient that the inequality

$$|G_1| \gg |G_2| + |H_2|, \quad |G_n| + \left| \sum_{m=1}^{n-1} H_{n,2m} \right| \gg |G_{n+1}| + \left| \sum_{m=1}^n H_{n+1,2m} \right|, \quad n \geq 2 \quad (1.26)$$

be satisfied.

The relations (1.25) and (1.26) together with (1.22)-(1.24) describe in a general form the sought approximation conditions. The latter depend on the transition instants as well as their functional dependence, and also on the type of the deviation of the distribution function from the equilibrium one.

The expressions are now given for the first three coefficients  $D_{n,2m}$ ,  $m = 1, 2, 3$ , and also a semiempirical formula to estimate the subsequent ones ( $n \geq m + 1$ ),

$$D_{n2} = \frac{n-1}{(2n)!}, \quad D_{n4} = -\frac{(n-1)}{(2n)!} \left[ \frac{n(2n-1)}{6} - 1 \right],$$

$$D_{n6} = \frac{n-1}{(2n)!} + \frac{1}{4!(2n-3)!} \left[ \frac{(2n-3)(2n-4)}{10} - 1 \right],$$

$$D_{n,2m} \cong \frac{n-1}{(2n)!} + \frac{1}{4!} \sum_{s=1}^{m-1} \frac{(-1)^s q^{s-1}}{(2n-2s-1)!}, \quad q \cong 0.1013, \quad m \geq 4;$$

the last formula follows from (1.24) if one adopts  $A_{2m}/A_{2m-2} = -q \approx \text{const}$ ; in fact, the ratio  $q$  varies very little, for example, from 0.10131 for  $m = 4$  to  $q = 0.10132$  for  $m = 20$ .

## 2. Approximation Conditions and Transition Instant Features for Rayleigh Gases

To give an example, the approximation conditions are analyzed of divergence flow by means of a single term in the case of relaxation of a comparatively "hot" Rayleigh gas, that is, in the case of a small admixture of heavy particles to the medium light atoms of the thermostat;  $m_2/m_1 = \lambda \ll 1$  ( $m_1$  is the particle mass,  $m_2$  is the atomic mass). It is assumed that the initial particle distribution with respect to energy is a Boltzmann one with temperature  $T_0$ , where  $T_0 \gg T$ , i.e.,

$$f(x, 0) = n_1 \frac{2}{\sqrt{\pi}} \frac{\sqrt{x}}{(kT_0)^{3/2}} e^{-\frac{x}{kT_0}}, \quad T_0 \gg T, \quad (2.1)$$

$x = m_1 v_1^2/2$ ,  $v_1$  is particle velocity,  $n_1 = \int_0^\infty f(x, 0) dx = \int_0^\infty f(x, t) dx$  is particle concentration. By confining our considerations to the initial essentially nonequilibrium relaxation stage distribution and by using (2.1)† for the instants  $t < \tau_r$  one may set

$$\psi = \frac{1}{f^0} \left( \frac{\partial f}{\partial x} - f \frac{\partial \ln f^0}{\partial x} \right) \cong \frac{\varphi}{kT}, \quad x > kT; \quad (2.2)$$

$$\partial^l \psi / \partial x^l \cong \psi / (kT)^l. \quad (2.3)$$

By changing in (1.20)-(1.23) to the variable  $y = x/kT$  and by using (2.3) one obtains‡

$$kT \frac{\partial f}{\partial t} = -\text{div } j(y); \quad (2.4)$$

$$-j(y) = \sum_{n=1}^{n_*} \left\{ G_n(y) + \sum_{m=1}^{n-1} H_{n,2m}(y) \right\}, \quad (2.5)$$

where

$$G_n(y) = \frac{1}{(2n)!} \frac{B_{2n}}{(kT)^{2n}} f^0(y) \psi(y), \quad \psi(y) = \frac{\partial \varphi(x, t)}{\partial y}; \quad (2.6)$$

$$H_{n,2m}(y) = D_{n,2m} \psi(y) \frac{d^{2m-1}}{dy^{2m-1}} \left( \frac{B_{2n}}{(kT)^{2n}} f^0(y) - \frac{d}{dy} \left( \frac{B_{2n}}{(kT)^{2n}} f^0(y) \right) \right). \quad (2.7)$$

The approximation conditions (1.25), (1.26) together with (2.6) and (2.7) depend on the properties  $B_{2n}$  of the transition instants (and on the thermostat temperature); in particular, (1.25) with  $f_0 \sim \sqrt{y} e^{-y}$  taken into account as well as (2.6) and (2.7) assumes the form

\*The initial distribution need not be a Boltzmann one; it is important, however, that the approximation (2.2) be valid.

†Since (2.2) and (2.3) are approximations we leave  $n_*$  terms in the sum with respect to  $n$ ; (1.20)-(1.23) is not strongly equivalent to (1.1) for the problem under consideration because of lack of symmetry in the integration limits; the latter can, however, be ignored for  $x > kT$  ( $x \gg 2\langle \Delta^2 \rangle$ ).

$$\frac{1}{2} \frac{B_2}{(kT)^2} \gg \left| \sum_{n=2}^{n_*} \left\{ \frac{1}{(2n)!} \frac{B_{2n}}{(kT)^{2n}} + \frac{1}{\sqrt{y}} \sum_{m=1}^{n-1} D_{n,2m} \sum_{s=0}^{2m-1} (-1)^{s+1} C_{2m-1}^s \frac{d^s}{dy^s} \left( \frac{2B_{2n} \sqrt{y}}{(kT)^{2n}} - \frac{d}{dy} \left( \frac{B_{2n} \sqrt{y}}{(kT)^{2n}} \right) \right) \right\} \right| \quad (2.8)$$

( $C_n^s$  are binomial coefficients); moreover, the first condition (1.26) becomes

$$\frac{1}{2} \frac{B_2}{(kT)^2} \gg \frac{1}{4!} \frac{B_4}{(kT)^4} + \frac{1}{4! \sqrt{y}} \left| \sum_{s=0}^1 (-1)^{s+1} \frac{d^s}{dy^s} \left( \frac{2B_4 \sqrt{y}}{(kT)^4} - \frac{d}{dy} \left( \frac{B_4 \sqrt{y}}{(kT)^4} \right) \right) \right|. \quad (2.9)$$

Now the coefficients  $B_{2n}$ , the transition instants for a Rayleigh gas, should be made more specific. The transition instant of the second order ( $n = 1$ ) has often been calculated when analyzing the relaxation of a Rayleigh gas. In [4]\* this instant was evaluated in the first approximation in the ratio of the masses of atom and of particle, the computation being carried out for the general form of the interaction potential and also the coefficients  $B_2$  being expressed in terms of quantities well known in the kinetic theory of gases. In [5] in the case of interaction exact values were obtained, according to the law of solid spheres for the first three transition instants,  $\langle \Delta \rangle$ ,  $\langle \Delta^2 \rangle$ , and  $\langle \Delta^3 \rangle$ ; these instants were evaluated using the analytic expression obtained in the same paper for the probability  $\psi W(x, \Delta)$  which permits, in principle, evaluation of instants of higher orders; however, for  $n > 1$  this would be too cumbersome.

It suffices for the purpose of this article to limit the computation to the same approximation as in [4]; this approximation corresponds to the first-order computations in the perturbation theory of the interactions for which the atom energy changes resulting from the collision with a particle can be disregarded. However, it was not specified in [4] what order of magnitude in  $\lambda$  ( $\sqrt{\lambda}$  or  $\lambda$ ) should have the rejected terms.

Below using the kinetic theory [6] the pattern of computing the coefficients  $B_{2n}$  is briefly described without using explicitly  $W(x, \Delta)$ , and it is shown that the exact evaluation of  $B_{2n}$  should contain terms with  $\lambda$  only of integer degree. Therefore, if in the first approximation in the expression for  $B_{2n}$  one takes into account the terms of the lowest degree  $\lambda \sim \lambda^s$ , then the neglected terms are proportional to  $\lambda^{k+s}$ ,  $k \geq 1$ . The values are thus obtained of  $B_{2n}$  in the first approximation. The approach described here enables one to evaluate the transition instants with any approximation in  $\lambda$ ; such evaluations, however, are outside the scope of this article.

To evaluate  $B_{2n}$  one has to analyze the collision dynamics between a particle and an atom. The following notation is introduced;  $\mathbf{v}_1$ ,  $\mathbf{p}_1$  are the particle velocity and momentum;  $\mathbf{v}_2$ ,  $\mathbf{p}_2$  are the atom velocity and momentum;  $\mathbf{g} = \mathbf{v}_2 - \mathbf{v}_1$ ;  $M = m_1 m_2 / (m_1 + m_2) = m_2 / (1 + \lambda)$ . The energy change in the particle resulting from a collision is equal to

$$\Delta x = \frac{(\Delta p_1)^2}{2m_1} + \frac{(\mathbf{p}_1 \Delta \mathbf{p}_1)}{m_1},$$

$\Delta \mathbf{p}_1 = -\Delta \mathbf{p}_2$ ,  $\Delta p_i$  is the change of momentum of the  $i$ -th particle due to collision,

$$(\Delta x)^{2n} = \left( \frac{1}{2m_1} \right)^{2n} \sum_{r=0}^{2n} (2m_1)^r C_{2n}^r (\Delta p_2)^{4n-r} (\mathbf{v}_1 \mathbf{k})^r, \quad \mathbf{k} = \frac{\Delta \mathbf{p}_1}{\Delta p_1}. \quad (2.10)$$

It is further assumed that the collision dynamics corresponds to the case of interaction according to the law of solid spheres [6]. Then one has  $\Delta p_2 = 2Mg \cos \psi_1$ ,  $\psi_1 = (1/2)(\pi - \chi)$ , where  $\chi$  is the scattering angle for collisions,

$$\frac{\langle (\Delta x)^{2n} \rangle}{\tau} = \int (\Delta x)^{2n} dN, \quad (2.11)$$

where  $dN$  is the number of atoms whose velocity lies in the interval  $\mathbf{v}_2$ ,  $\mathbf{v}_2 + d\mathbf{v}_2$  scattered by the particle at an angle  $\chi$ ,  $\chi + d\chi$  in a unit of time,

$$dN = g f_2^0(\mathbf{v}_2) dv_2 d\sigma,$$

where  $d\sigma$  is the scatter section at an angle  $\chi$ ,  $\chi + d\chi$ ;  $f_2^0 = n_2 (m_2 / 2\pi kT)^{3/2} e^{-m_2 v_2^2 / 2kT}$ ;  $n_2$  is the concentration of atoms; for the interaction model under consideration one has

\*In fact, in [4] a wider problem was considered, namely, the relaxation of Rayleigh gas with rotatory degrees of freedom; in the approximation analyzed in [4] the translational and rotational relaxations proceed independently.

$$dN = g\sigma_{12}^2 \cos \psi_1 \sin \psi_1 f_2^0 v_2^2 \sin \theta d\theta d\psi_1 d\epsilon d\varphi dv_2; \quad (2.12)$$

in the above  $\psi_1$  and  $\epsilon$  are the orbital and azimuthal angles of the vector  $\mathbf{k}$  relative to  $\mathbf{g}$ , and  $\theta$  and  $\varphi$  are the corresponding angles of the vector  $\mathbf{v}_2$  relative to  $\mathbf{v}_1$ ;  $\psi_1$  varies from 0 to  $\pi/2$ ;  $\theta$  from 0 to  $\pi$ ;  $\epsilon$  and  $\varphi$  from  $-\pi$  to  $2\pi$ ;  $\sigma_{12} = (\mathbf{1}/2)(\sigma_1 + \sigma_2)$ ,  $\sigma_1$  is the diameter of the  $i$ -th sphere.

In (2.12) one can change from the integration variable  $\theta$  to  $g$ :

$$g^2 = v_1^2 + v_2^2 - 2v_1v_2 \cos \theta, \quad gdg = v_1v_2 \sin \theta d\theta,$$

the integration limits with respect to  $dg$  being from  $v_2 - v_1$  to  $v_2 + v_1$  for  $v_2 > v_1$  and from  $v_1 - v_2$  to  $v_2 + v_1$  for  $v_2 < v_1$ .

If one takes into account (2.10) and (2.12) the quantity  $(\mathbf{v}_1\mathbf{k})$  remains undetermined in (2.11); by codirecting the  $z$  axis with the vector  $\mathbf{g}$  one obtains

$$(\mathbf{v}_1\mathbf{k}) = v_1 \sin \alpha \sin \psi_1 \cos(\varphi - \epsilon) + v_1 \cos \psi_1 \cos \alpha. \quad (2.13)$$

By using  $g^2 = v_1^2 + v_2^2 - 2(\mathbf{v}_1(\mathbf{g} + \mathbf{v}_1)) = v_2^2 - v_1^2 - 2v_1g \cos \alpha$  one finds

$$\cos \alpha = \frac{1}{2gv_1} (g^2 + v_1^2 - v_2^2) \equiv \frac{A_1}{2gv_1}, \quad (2.14)$$

the relation (2.14) expressing  $(\mathbf{v}_1\mathbf{k})$  by means of the variables  $g$ ,  $v_2$ ,  $\psi_1$ ,  $\epsilon$ ,  $\varphi$ .

Thus, one obtains from (2.11) together with (2.10), (2.12), and (2.13) that

$$B = \left(\frac{1}{2m_1}\right)^{2n} \sigma_{12}^2 \sum_{r=0}^{2n} \sum_{s=1}^{\left[\frac{r}{2}\right]} K_{nr} (2m_1)^r \int_0^{v_2+v_1} f_2^0 g^{4n-r} v_1^r \cos^{r-2s} \alpha \sin^{2s} \alpha \frac{v_2}{v_1} dv_2 dg, \quad (2.15)$$

[ $r/2$ ] is the integral part of  $r/2$ ,

$$K_{nr} = C_{2n}^r C_r^s \int \cos^{4n-2s+1} \psi_1 \sin^{2s+1} \psi_1 \cos^{2s}(\varphi - \epsilon) d\psi_1 d\epsilon d\varphi.$$

It follows from the integral in (2.15) that it can be represented in the form (the superscripts  $l$ ,  $k$ , and  $m$  being understood formally)

$$v_1^{2k} \int_0^{v_2+v_1} f_2^0 g^{2l} \frac{v_2}{v_1} v_2^{2m} dv_2 dg = v_1^{2k} \left\{ \frac{1}{v_1} \int_{v_1}^{\infty} e^{-\frac{m_2 v_2^2}{2kT}} v_2^{2m+1} \int_{v_2-v_1}^{v_2+v_1} g^{2l} dg dv_2 + \frac{1}{v_1} \int_0^{v_1} e^{-\frac{m_2 v_2^2}{2kT}} v_2^{2m+1} \int_{v_1-v_2}^{v_2+v_1} g^{2l} dg dv_2 \right\},$$

which can be reduced to the integrals

$$\int_{\lambda y}^{\infty} e^{-z} z^k dz \text{ and } \frac{1}{V^{\lambda y}} \int_0^{V^{\lambda y}} e^{-z^2} dz, \quad y = \frac{x}{kT},$$

which are rational functions of  $\lambda y$ , that is, contain only  $(\lambda y)^k$ ,  $k \geq 0$ . Thus,  $B_{2n}$  contains finally terms with  $\lambda^k$ ,  $k \geq 1$ , only. The expression (2.15) together with (2.10) yields exact values of  $B_{2n}$ .

Let us consider an instant of the zeroth order,  $-1/\tau$ , which corresponds to the collision frequency  $P_{12}(v_1)$  of a particle of velocity  $v_1$  with atoms and equal to [6]

$$\frac{1}{\tau} = 2n_2 \sigma_{12}^2 \left(\frac{2\pi kT}{m_2}\right)^{1/2} \left\{ e^{-\lambda y} + (2\lambda y + 1) \int_0^{V^{\lambda y}} e^{-z^2} dz \right\} = \frac{1}{\tau_0} + 0(\lambda y)$$

( $\tau_0$  being the free path of the particle in the gas).

The sought values of  $B_{2n}$  can be obtained in the first approximation by setting  $g \approx v_2$ ,  $(\mathbf{v}_1\mathbf{k}) \approx v_1 \cos \gamma$  in (2.11) together with (2.10) and (2.12) and averaging  $\cos^2 \gamma$  over the sphere; this results in

$$B_{2n} = \frac{1}{\tau_0} b_{2n} y^n (kT)^{2n} [1 + 0(\lambda^s, (\lambda y)^l)], \quad (2.16)$$

$$s, l \geq 1, b_{2n} = \frac{n! (16\lambda)^n}{2(2n+1)}, \quad y \gg n(n+1)(2n+1)\lambda.$$

By using (2.16) the condition (2.8) becomes

$$\frac{1}{2} b_2 y \gg \left| \sum_{n=2}^{n_*} \frac{1}{(2n)!} b_{2n} y^n \left\{ 1 + (2n)! \sum_{m=1}^{n-1} D_{n,2m} \sum_{s=0}^{2m-1} (-1)^{s+1} C_{2m-1}^s \frac{h_{ns}}{y^s} \left( 2 - \frac{2n-2s+1}{2y} \right) \right\} \right|, \quad (2.17)$$

where it was assumed that  $(d^s y^{n+1/2})/dy^s \equiv h_{ns} y^{n+(1/2)-s}$ ; (2.17) is not strictly equivalent to (2.8) since each  $n$ -th term of the series in (2.17) is determined with an accuracy up to  $\lambda^r$ ,  $r \geq n+1$ . However, this relation shows that if on the right of (2.17) a considerable contribution is given by the terms with  $n \leq n_*$ , then for a strict determination of necessary conditions for the approximation at the initial stage of the procedure one has to employ in (2.8) a value of  $B_{2n}$  evaluated with an accuracy up to the terms  $\sim \lambda^{n_*+1}$ .

The evaluation of  $B_{2n}$  in (2.16) enables one to determine sufficient conditions of the approximation as well as to estimate the order of magnitude of the terms appearing in the divergence flow.

These sufficient conditions for approximating the divergence flow by a single term [see (1.26)] correspond now to a requirement that the  $n$ -th term of the series in (2.17), in which the second term was taken in its modulus, be much bigger than the next  $(n+1)$ -th term; the latter conditions enable one to obtain relations between the possible values of  $y$  and  $\lambda$ ; in particular, one can obtain from them for the upper bound of the values of  $y$  ( $y \gg 1$ ) for  $n \geq 5$ ,  $n^3 < y/\lambda$  that  $y \ll 1/16\lambda$  follows. If the considerations are limited to the condition (2.9) only together with (2.16) then one finds that the Fokker-Planck term is much bigger than the next one for  $y \ll 1/\lambda$ .

It also follows from (2.4)-(2.7) together with (2.16) that each subsequent term in the divergence flow relative to the preceding one is of the order of  $\sim \lambda$  and in the equation of Fokker-Planck type of coefficient  $B_2$  is, as a rule, sufficient for evaluating with an accuracy up to  $\lambda$ ,  $B_2 = (8/3\tau_0)\lambda y(kT)^2$ , since the neglected terms in the equation are of the order  $B_{2n} \sim \lambda^n$ ,  $n \geq 2$ .

#### LITERATURE CITED

1. A. Siegel, "Differential-operator approximations to the linear Boltzmann equation," J. Math. Phys., 1, No. 2, 378 (1960).
2. N. J. Van Kampen, "A power series expansion of the master equation," Can. J. Phys., 39, No. 4, 551 (1961).
3. H. Akama and A. Siegel, "Small-parameter expansions of linear Boltzmann collision operators," Physica, 31, No. 10, 1493 (1965).
4. M. N. Safaryan and E. V. Stupochenko, "Rotary relaxation of two-atom molecules in a light inert gas," Zh. Prikl. Mekh. Tekh. Fiz., No. 4 (1964).
5. K. Andersen and K. Shuler, "On the relaxation of the hardsphere Rayleigh and Lorentz gas," J. Chem. Phys., 40, No. 3, 633 (1964).
6. S. Chapman and T. Cowling, Mathematical Theory of Non-Uniform Gases, 3rd ed., Cambridge University Press (1970).